Journal of Sound and Vibration (1998) **210**(2), 279–285

LETTERS TO THE EDITOR

COMMENTS ON "RESPONSE ERRORS OF NON-PROPORTIONALLY LIGHTLY DAMPED STRUCTURES"

S. M. SHAHRUZ

Berkeley Engineering Research Institute, *P.O. Box* 9984, *Berkeley*, *California* 94709, *U.S.A.*

(*Received* 15 *July* 1997)

In reference [1], the authors consider an n-degree-of-freedom linear system represented in the modal co-ordinates by

$$\ddot{q}(t) + C\dot{q}(t) + \mathbf{\Omega}^2 q(t) = Bf(t), \tag{1}$$

for all $t \geqslant 0$, where $q(t) \in \mathbb{R}^n$ is the vector of modal displacements, $f(t) \in \mathbb{R}$ is the input function, $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ is the modal damping matrix, $\Omega = diag[\omega_1, \omega_2, \ldots, \omega_n]$ is the matrix of natural frequencies, and the vector

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n, \tag{2}$$

describes the distribution of the applied input in the modal co-ordinates.

The authors decompose the matrix C as

$$C = C_d + C_o, \tag{3}$$

where $C_d = diag[c_{11}, c_{22}, \ldots, c_{nn}]$ and $C_o = [c_{oij}] \in \mathbb{R}^{n \times n}$ is a matrix with zero diagonal elements and $c_{oij} = c_{ij}$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$. The matrix C_d is written as

$$C_d = diag[2\xi_1\omega_1, 2\xi_2\omega_2, \ldots, 2\xi_n\omega_n], \tag{4}$$

where ξ_i is the ith modal damping ratio, which is positive for all $i = 1, 2, \ldots, n$.

The authors make two assumptions regarding the natural frequencies and the modal damping ratios:

(A1) The modal frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are not clustered.
(A2) The modal damping ratio $\xi_i \ll 1$ for all $i = 1, 2, \ldots, n$.

As is customary, the authors neglect the off-diagonal elements of C in equation (1) to obtain

$$\ddot{q}_p(t) + C_d\dot{q}_p(t) + \mathbf{\Omega}^2 q_p(t) = Bf(t), \tag{5}$$

for all $t \geqslant 0$, where $q_p(t) \in \mathbb{R}^n$. The system (5) is a set of n decoupled second order systems, which can be readily solved for $q_p(\cdot)$. The vector $q_p(\cdot)$ furnishes an approximate solution for $q(\cdot)$ in equation (1). The question is: Under what conditions is $q_p(\cdot)$ a reasonably accurate approximation of $q(\cdot)$? More precisely, define the error between the ith components of the vectors $q(\cdot)$ and $q_p(\cdot)$ by

$$e_i(t) := q_i(t) - q_{pi}(t) \in \mathbb{R}^n, \tag{6}$$

0022–460X/98/070279 + 07 $25.00/0/sv971272

for all $t \geqslant 0$. Under what conditions is $e_i(\cdot)$ small for all $i = 1, 2, \ldots, n$? The authors claim that if assumptions (A1) and (A2) hold, then $q(\cdot)$ and $q_p(\cdot)$ are close to each other. This claim is *not* true, as we will show in the following.

First, let us briefly review the argument given by the authors. In order to justify their claim, the authors introduce a scalar quantity, called the modal coupling, as

$$\kappa_{i,k} := \frac{|b_i|/|b_k|}{(\xi_i/\xi_k)((\omega_i/\omega_k)^2 + [(\omega_i/\omega_k)^2 - 1]^2/4\xi_i^2)^{1/2}}, \tag{7}$$

for all $i, k = 1, 2, \ldots, n$ and $i \neq k$, where b_i and ξ_i are, respectively, the ith element of the vector B in equation (2) and the ith modal damping ratio in equation (4). The authors define two more quantities as

$$\kappa_i := \max_{\substack{1 \leqslant k \leqslant n \\ k \neq i}} (\kappa_{i,k}), \qquad \sigma_i = \sum_{j=1}^{n} |c_{oij}|, \tag{8a, b}$$

for all $i = 1, 2, \ldots, n$, where c_{oij}, $j = 1, 2, \ldots, n$, are the off-diagonal elements of the ith row of the matrix C. Finally, the authors define the factor

$$s_i := \kappa_i \sigma_i / 2\xi_i \omega_i, \tag{9}$$

for all $i = 1, 2, \ldots, n$. Having the factor s_i defined, the authors erroneously show that if assumptions (A1) and (A2) hold, then

$$\frac{\|e_i\|_2}{\|q_i\|_2} \leqslant s_i \ll 1, \tag{10}$$

for all $i = 1, 2, \ldots, n$, where

$$\|q_i\|_2 := \int_0^\infty |q_i(t)|^2 \, \mathrm{d}t, \tag{11}$$

is the L_2-norm of the time function $t \mapsto q_i(t)$. From inequality (10), the authors conclude that the error in the approximate solution is small and hence "the off-diagonal elements of the damping matrix can be neglected regardless of their values".

The fact is that inequality (10) does *not* necessarily hold: the factor s_i can possibly be close to or even larger than 1 for some $i = 1, 2, \ldots, n$. Recall that s_i is given by equation (9). In this equation, it can be true that $\sigma_i/2\xi_i\omega_i \approx 1$ for all $i = 1, 2, \ldots, n$, however, κ_i defined originally in equation (8a) is not necessarily much less than 1. The authors reach the erroneous conclusion that $\kappa_{i,k}$ in equation (7) and κ_i are much smaller than 1 for all $i, k = 1, 2, \ldots, n$ and $i \neq k$, because they consider $\xi_i/\xi_k = 1$ and $|b_i|/|b_k| = 1$ for all $i, k = 1, 2, \ldots, n$ and $i \neq k$. It is true that if assumptions (A1) and (A2) hold and if $\xi_i/\xi_k = 1$ and $|b_i|/|b_k| = 1$ for all $i, k = 1, 2, \ldots, n$ and $i \neq k$, then $\kappa_{i,k} \ll 1$ and $\kappa_i \ll 1$ for all $i, k = 1, 2, \ldots, n$ and $i \neq k$. However, there is no reason to believe that $|b_i|/|b_k| = 1$ for all $i, k = 1, 2, \ldots, n$ and $i \neq k$, because the magnitudes of the elements of the vector B in equation (2) are not necessarily close or equal to each other. Since b_i can have any magnitude for different $i = 1, 2, \ldots, n$, it is possible to have $\kappa_{i,k}$ and κ_i close to or even larger than 1 for some $i, k = 1, 2, \ldots, n$ and $i \neq k$. Thus, s_i is not necessarily much smaller than 1 for all $i = 1, 2, \ldots, n$ and the approximation error may not be small. We will give an example to show that κ_i and s_i are large for some $i = 1, 2, \ldots, n$, and so is the approximation error.

The authors make a mistake in their Examples 1 and 2, when they use inequality (10) for functions that do not belong to the space $L_2(\mathbb{R}_+)$. In these examples, they apply step inputs to two systems. Due to the linearity of the systems and the positive definiteness of their modal damping matrices, the response $q_i(t)$ converges to a constant value q_i^* as $t \to \infty$ for all $i = 1, 2, \ldots, n$. That is, the steady state responses are step functions. Therefore, $q_i \notin L_2(\mathbb{R}_+)$ for all $i = 1, 2, \ldots, n$, the norm $\|q_i\|_2$ is meaningless ($\|q_i\|_2 = \infty$), and inequality (10) is not applicable. The use of the L_2-norm of time functions limits the application of inequality (10) to only those systems the responses of which belong to $L_2(\mathbb{R}_+)$. Thus, for instance, when the system responses are sinusoidal or step functions, inequality (10) is not applicable.

It should be pointed out that the modal coupling factor $\kappa_{i,k}$ defined in equation (7) fails to incorporate the frequency content of the applied input for all $i, k = 1, 2, \ldots, n$ and $i \neq k$. The usefulness of the modal coupling factors in providing a reasonable estimate for the approximation error is questionable, because they lack information regarding the input frequency. It is known that the input frequency can have a significant effect on the size of the approximation error, when it is close to one of the natural frequencies of the system [2]. We will see the significance of the input frequency in our example.

Now, we give an example of a system for which assumptions (A1) and (A2) hold. However, the approximate decoupling of this system by neglecting the off-diagonal elements of the modal damping matrix leads to inaccurate approximate solutions.

Consider a system represented in the modal co-ordinates by

$$\begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \ddot{q}_3(t) \end{bmatrix} + \begin{bmatrix} 0\cdot04 & -0\cdot02 & -0\cdot02 \\ -0\cdot02 & 0\cdot4 & -0\cdot08 \\ -0\cdot02 & -0\cdot08 & 0\cdot6 \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} \sin t, \quad (12)$$

for all $t \geqslant 0$. For this system,

$$\omega_1 = 1, \qquad \omega_2 = 5, \qquad \omega_3 = 10, \quad (13a)$$

$$\xi_1 = 0\cdot02, \qquad \xi_2 = 0\cdot04, \qquad \xi_3 = 0\cdot03, \quad (13b)$$

$$b_1 = 10, \qquad b_2 = 1, \qquad b_3 = 5. \quad (13c)$$

Clearly, assumptions (A1) and (A2) hold for the system (12). Therefore, according to the authors $\kappa_i \ll 1$ and $s_i \ll 1$ for all $i = 1, 2, 3$ and "the off-diagonal elements of the damping matrix can be neglected regardless of their values". We show that these conclusions are not true.

We now compute κ_i and s_i for $i = 1, 2, 3$. From equation (7), we obtain

$$\kappa_{1,2} = 0\cdot8333, \qquad \kappa_{1,3} = 0\cdot1212, \quad (14a)$$

$$\kappa_{2,1} = 0\cdot00016, \qquad \kappa_{2,3} = 0\cdot01598, \quad (14b)$$

$$\kappa_{3,1} = 0\cdot0002, \qquad \kappa_{3,2} = 0\cdot1332. \quad (14c)$$

Thus, from equations (8) and (9), we obtain

$$\kappa_1 = 0\cdot8333, \qquad \kappa_2 = 0\cdot01598, \qquad \kappa_3 = 0\cdot1332, \quad (15a)$$

$$s_1 = 0\cdot8333, \qquad s_2 = 0\cdot0039, \qquad s_3 = 0\cdot0222. \quad (15b)$$

Obviously, s_1 is not much smaller than 1. This is due to the fact that b_1, b_2 and b_3 have different magnitudes.

We next study the exact and approximate solutions of the system (12) as well as the approximation error. The approximate solution is the solution of the following decoupled system which is obtained by neglecting the off-diagonal elements of the modal damping matrix in equation (12):

$$\begin{bmatrix} \ddot{q}_{p1}(t) \\ \ddot{q}_{p2}(t) \\ \ddot{q}_{p3}(t) \end{bmatrix} + diag[0{\cdot}04,\ 0{\cdot}4,\ 0{\cdot}6] \begin{bmatrix} \dot{q}_{p1}(t) \\ \dot{q}_{p2}(t) \\ \dot{q}_{p3}(t) \end{bmatrix} + diag[1,\ 25,\ 100] \begin{bmatrix} q_{p1}(t) \\ q_{p2}(t) \\ q_{p3}(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} \sin t, \quad (16)$$

for all $t \geqslant 0$. We solved the systems (12) and (16) numerically for the vectors $[q_1(\cdot)\ q_2(\cdot)\ q_3(\cdot)]^{\mathrm{T}}$ and $[q_{p_1}(\cdot)\ q_{p_2}(\cdot)\ q_{p_3}(\cdot)]^{\mathrm{T}}$, respectively. In Figures 1, 2 and 3, we have plotted $t \mapsto q_i(t)$, $t \mapsto q_{p_i}(t)$ and $t \mapsto e_i(t) = q_i(t) - q_{pi}(t)$, respectively, for $i = 1, 2, 3$. For the second mode, we have plotted the exact and approximate steady state solutions and their difference in Figure 4, and have denoted them, respectively, by $t \mapsto q_2^s(t)$, $t \mapsto q_{p2}^s(t)$ and $t \mapsto e_2^s(t)$.

We note that due to the linearity of the systems (12) and (16), the exact and approximate steady state solutions are sinusoidal functions of time. Thus, $q_i \notin L_2(\mathbb{R}_+)$ for all $i = 1, 2, 3$ and $\|q_i\|_2$ cannot be computed. Therefore, inequality (10) is not applicable. We also note that the approximation error $e_2(\cdot)$ is quite large. Hence, the authors' claim that "the off-diagonal terms rarely cause a significant approximation error when the damping is small" is not true. The off-diagonal elements of the modal damping matrix *do* cause a significant error. Roughly speaking, the reason for the large $e_2(\cdot)$ is as follows. Note that the damping ratio for the first mode is small, the frequency of the applied input is equal to the natural frequency $\omega_1 = 1$, and the amplitude of the input to the first mode is large. Therefore, the first mode is in resonance and the steady state solution for this mode has the large amplitude of 250, as is evident in Figure 1. Due to this large amplitude, any coupling of the first mode to the other modes through an off-diagonal element of the modal matrix can have a significant effect on the other modes. Thus, if the off-diagonal elements of the modal damping matrix are neglected, the significant effect of the first mode on the

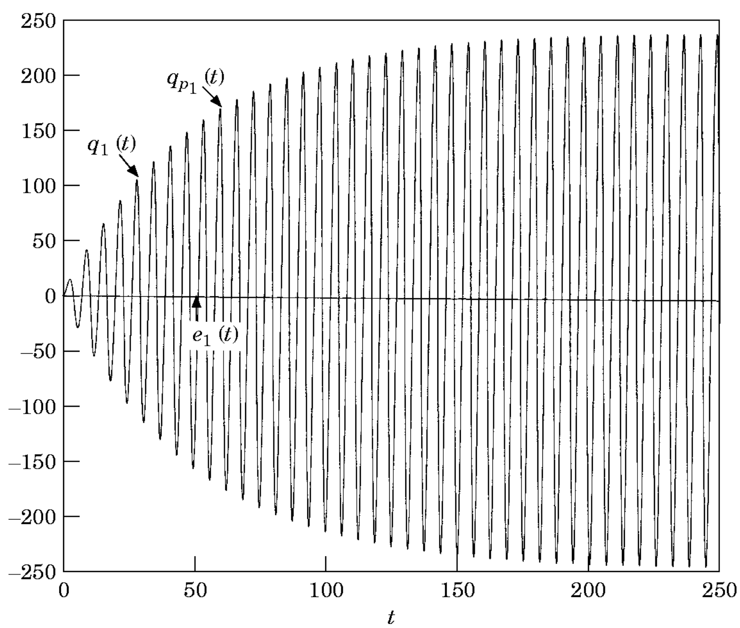

Figure 1. The exact and approximate solutions for the first mode and the approximation error denoted by $q_1(\cdot)$, $q_{p_1}(\cdot)$ and $e_1(\cdot)$, respectively. The error $e_1(\cdot)$ is almost zero.

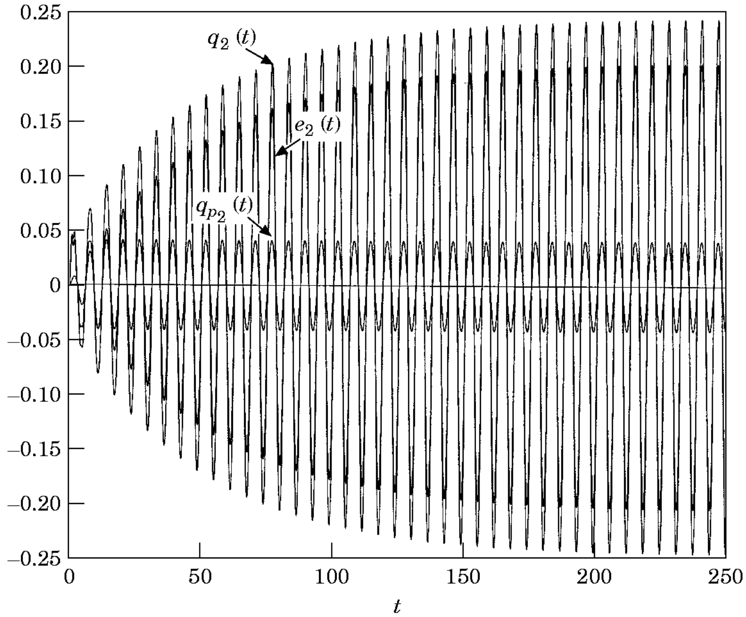

Figure 2. The exact and approximate solutions for the second mode and the approximation error denoted by $q_2(\cdot)$, $q_{p_2}(\cdot)$ and $e_2(\cdot)$, respectively. The error $e_2(\cdot)$ is quite large.

other modes will be eliminated and, consequently, there will be a large error between the exact and approximate solutions. Note also that our example shows the important role of the input frequency in causing resonance in one of the modes and a large approximation error.

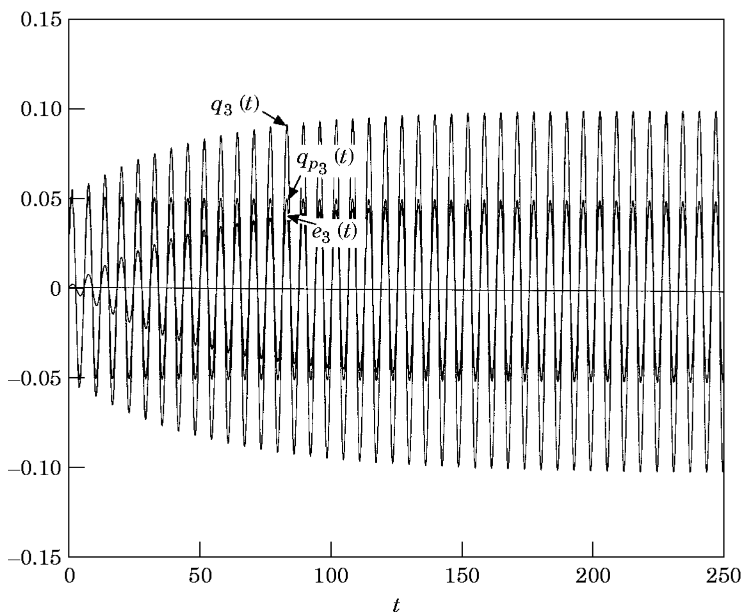

Figure 3. The exact and approximate solutions for the third mode and the approximation error denoted by $q_3(\cdot)$, $q_{p3}(\cdot)$ and $e_3(\cdot)$, respectively. The error $e_3(\cdot)$ is large.

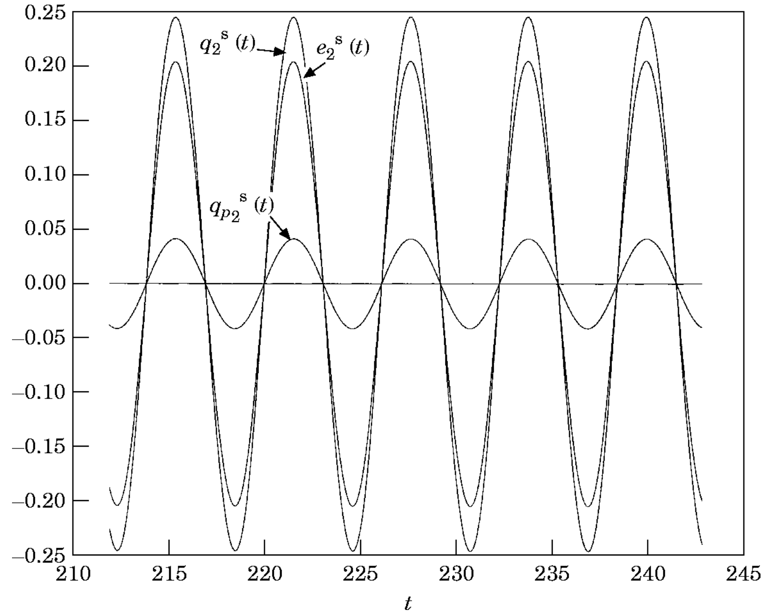

Figure 4. The exact and approximate steady state solutions for the second mode and their difference denoted by $q_2^s(\cdot)$, $q_{p_2}^s(\cdot)$ and $e_2^s(\cdot)$, respectively.

In a recent paper [3], it is shown that the size of the approximation error in the modal (also called normalized) co-ordinates does not provide a definitive measure of the size of the error in the physical co-ordinates, where the physical variables of interest are measured. For instance, it is possible to have small errors in the modal co-ordinates, but have large errors in the physical co-ordinates. Therefore, it is futile to seek criteria that guarantee a small approximation error in the modal co-ordinates. The modal analysis can certainly be used to obtain useful information regarding the natural frequencies and the modes of systems. However, using the modal matrix in order to transform the system to the modal co-ordinates and to obtain an approximate solution for the system is not a good approach, because the accuracy of the approximate solution in the physical co-ordinates remains unknown.

REFERENCES

1. W. Gawronski and J. T. Sawicki 1997 *Journal of Sound and Vibration* **200**, 543–550. Response errors of non-proportionally lightly damped structures.
2. S. M. Shahruz and A. K. Packard 1993 *Transactions of the American Society of Mechanical Engineers*, *Journal of Dynamic Systems*, *Measurement*, *and Control* **115**, 214–218. Approximate decoupling of weakly nonclassically damped linear second-order systems under harmonic excitations.
3. S. M. Shahruz and P. A. Srimatsya 1997 *Journal of Sound and Vibration* **201**, 262–271. Approximate solutions of non-classically damped linear systems in normalized and physical coordinates.

AUTHORS' REPLY

W. Gawronski

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109, *U.S.A.*

AND

J. T. Sawicki

Department of Mechanical Engineering, Cleveland State University, Cleveland, Ohio 44115, *U.S.A.*

(*Received* 18 *August* 1997)

The authors wish to thank Dr Shahruz for his comments [1] on the authors' letter [2], in which he questioned: (1) that the assumptions of $|b_k|/|b_i| = 1$ and $|\xi_K|/|\xi_i| = 1$ lead to the erroneous conclusions regarding the output error; (2) that the two-norm was mistakenly used for the step response; and (3) that the error can exceed the output, as shown in the example of [1].

Regarding the first question. In our letter [2] we did not explain that the assumption $|b_k|/|b_i| = 1$ denotes the worst case of equation (7). Indeed, consider $|b_k|/|b_i| \ll 1$. In this case the kth mode is excited lightly, and the kth error is negligible at the output. Similarly, the ratio $|\xi_k|/|\xi_i| = 1$ denotes the worst case in equation (7) of reference [2]. Certainly, for $|\xi_k|/|\xi_i| \ll 1$ the ith mode is excited lightly, and the ith error is negligible at the output.

In order to explain the second question, note that the two-norm was finite in our example. We used the two-norm of the step response in a limited time segment, up to the moment when the motion is stationary. In our case it was from 0 to 10 s in Example 1, and from 0 to 50 s in Example 2, as in Figure 4 of reference [2].

Now we turn to the last question. The example of reference [1] shows that errors under specific conditions are substantial. The magnitude of the error could be considered small or large, depending on the signal it is compared to. In the discussed example, the frequency of the harmonic excitation was equal to the first resonance frequency (of 1 rad/s). Therefore the first mode response is dominant (of amplitude 250), while the responses of the remaining modes are negligible (of amplitudes 0·25 and 0·1). The output as a combination of all modal responses is dominated by the first mode response in the example. The errors of the first mode, as well as the errors of the remaining modes, are negligible when compared to the output (less than 0·1%).

Finally, the claim in reference [1] that "the error in modal co-ordinates does not provide a definitive measure of size in the physical co-ordinates" is not true if the output error is considered: output does not depend on a choice of co-ordinates.

In conclusion, we consider an output error as a measure of the system performance. We thank Dr Shahruz for pointing out the inconsistencies in explaining the problem assumptions and in conditions of norm computation.

REFERENCES

1. S. M. Shahruz 1998 *Journal of Sound and Vibration* **210,** 279–284. Comments on "Response errors of non-proportionally lightly damped structures".
2. W. Gawronski and J. T. Sawicki 1997 *Journal of Sound and Vibration* **200**, 543–550. Response errors of non-proportionally lightly damped structures.